

Determining structure of real-space entanglement spectrum from approximate conditional independence

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We study ground state of gapped quantum many-body systems whose entanglement entropy S_A can be expressed as $S_A = a|\partial A| - \gamma$, where a, γ are some constants and $|\partial A|$ is an area of subsystem A . By using the structure of states that are conditionally independent, we argue that certain linear combination of real-space entanglement spectrum has small correlation with almost any local operator. We propose an information-theoretic conjecture under which the aforementioned statement can be established even if the formula for entanglement entropy holds approximately. The conjecture is an extension of strong subadditivity of entropy.

It is commonly believed that gapped phases of quantum many-body systems exhibit area law: entanglement entropy of a simply connected subsystem increases as the area of the boundary.[1] Overwhelming amount of evidences supporting this statement has been suggested, including explicit proof for ground state of 1D gapped system[2], exactly solvable models[3], and variational wavefunctions[4]. Constant subcorrection to the entanglement entropy - known as topological entanglement entropy - can be extracted by judiciously choosing a set of subsystems that cancels out the boundary contributions.[5, 6] Topological entanglement entropy is believed to be a universal constant characterizing the phase of the quantum many-body system.

Li and Haldane(LH) were first to realize that the spectrum of reduced density matrix may reveal extra information about the phase.[7, 8] While LH studied reduced density matrix in the orbital cuts, one may study its spectrum along real-space partition and arrive at a similar conclusion.[9] In particular, it has been recently suggested by several authors that entanglement spectrum along a real-space partition has a low-lying part that can be described by a local field theory.[10–12]

Topological entanglement entropy can be obtained from real-space entanglement spectrum of variational wavefunctions, similar to the way it is extracted from entanglement entropy.[12] Consequently the corresponding linear combination of entanglement spectrum is ‘topological’, in a sense that i) it does not interact with any local observable ii) it is equal to topological entanglement entropy.

Here we claim that existence of such topological operator can be attributed to approximate conditional independence of these quantum states. A tripartite state ρ_{ABC} is conditionally independent if conditional mutual information $I(A : C|B) = S_{AB} + S_{BC} - S_B - S_{ABC}$ is equal to 0. State is approximately conditionally independent if 0 is replaced by a small number $\epsilon > 0$. To the best of author’s knowledge, Hastings and Poulin were the first to point out that there can be configurations that are conditionally independent even in a quantum many-

body system with long range entanglement.[13] To illustrate their idea, suppose entanglement entropy satisfies area law with constant subcorrection term.

$$S_A = a|\partial A| - \gamma, \quad (1)$$

One can show that $I(A : C|B) = 0$ for choice of A, B, C such that i) AB, BC, B, ABC are all simply connected ii) A and C do not share a boundary.

State that is conditionally independent saturates the equality condition of strong subadditivity of entropy.[14] Such state forms a quantum Markov chain, and the structure of density matrix is vastly restricted compared to an arbitrary state.[15] In a form that is relevant to the present work, Petz showed states that are conditionally independent must satisfy the following relation.[16]

$$\hat{H}_{AB} + \hat{H}_{BC} - \hat{H}_B - \hat{H}_{ABC} = 0, \quad (2)$$

where $\hat{H}_A = -I_{A^c} \otimes \log \rho_A$ is a formal definition of entanglement spectrum. From now on, we denote the left hand side of the equation as $\hat{H}_{A:C|B}$ and refer to it as a *conditional mutual spectrum* of ABC . It follows that

$$\mathcal{C}(\hat{H}_{A:C|B}, X) = 0, \quad (3)$$

where $\mathcal{C}(\hat{H}_{A:C|B}, X) = \langle \hat{H}_{A:C|B} X \rangle - \langle \hat{H}_{A:C|B} \rangle \langle X \rangle$ is a connected correlation function between $\hat{H}_{A:C|B}$ and X . $\langle \dots \rangle$ denotes ground state expectation value.

While such operator trivially has zero correlation with any local operator, exact conditional independence is rarely satisfied for any realistic physical system. For example, if area law is violated with small correction, one cannot use eq.2 anymore. Even if area law holds exactly, eq.2 still cannot answer whether $\hat{H}_{A:C|B}$ is a topological operator for topologically nontrivial configuration, i.e. $I(A : C|B) \neq 0$.

These observations motivate a nontrivial statement about the spectrum of reduced density matrices. We conjecture following form of operator inequality to hold for general quantum states.

Conjecture 1.

$$\text{Tr}_{BC}(\rho_{ABC} \hat{H}_{A:C|B}) \geq 0. \quad (4)$$

Notice that the inequality implies strong subadditivity of entropy. One can see this by taking a partial trace over A . We shall first derive the consequence of this conjecture, and then provide the evidences.

Consequence of the conjecture— Brief comment on the notation is in order. For a conditional mutual spectrum $\hat{H}_{A:C|B}$, we shall refer B as the *reference party* and A, C as *target party*. Also, we shall diagrammatically represent the operator $\hat{H}_{A:C|B}$ with the following rule. The reference party corresponds to the region with ‘R’ sign. Each of the target parties corresponds to one of the simply connected regions with ‘T’ sign. When taking a partial trace, subsystem X is used to denote the nontrivial support of operator X . Shaded region in the diagram is a nontrivial support of X .

We postulate following modified form of entanglement entropy scaling relation to account for the deviations from ideal area law.

$$S_A = a|\partial A| - \gamma + \epsilon_A. \quad (5)$$

$$S_A + S_B - S_{AB} = \epsilon_{A:B}. \quad (6)$$

For large enough subsystem size, we expect ϵ_A to approach 0. $\epsilon_{A:B}$ denotes a long range correlation of the ground state. Due to exponential clustering theorem, we expect $\epsilon_{A:B}$ to scale as $\min(|A|, |B|)^2 e^{-\frac{2l}{\xi}}$, where ξ is a correlation length and $|A|$ is a volume of subsystem A . [27]

To simplify the analysis, we assume that each of the subsystems are sufficiently smooth and their boundary length is $O(l)$. We assume that support of X is sufficiently small compared to the size of the subsystems. We also assume that X is supported on one of the subsystems $A, B, C, D = (ABC)^c$.

The key idea for generalizing eq.3 is that one can decompose $\hat{H}_{A:C|B}$ into a sum of $\hat{H}_{A_i:C_i|B_i}$ in such a way that either i) $I(A_i : C_i|B_i)$ is small or ii) $A_i B_i C_i$ is sufficiently far away from support of X . Such decomposition can be derived from a simple application of chain rule.

$$\hat{H}_{A_1 A_2 : C|B} = \hat{H}_{A_2 : C|B} + \hat{H}_{A_1 : C|A_2 B}, \quad (7)$$

which can be verified from the definition of $\hat{H}_{A:C|B}$. While any deformation of subsystem can be expressed as a linear combination of chain rule, we define three elementary deformation moves for the clarity of exposition.

First example is *isolation move*. Goal of isolation move is to deform the boundary between the target party and reference party so that X can be separated from the reference party with distance $O(l)$. See FIG.1 We also define *separation move*. Purpose of the separation move is to deform the target party so that X is sufficiently separated from the target party. See FIG.2.

By first applying isolation move and then separation move, one can always deform the configuration to be distance $O(l)$ from X . The correction from the deformation is of the form $\text{Tr}(\rho_{A_i B_i C_i} \hat{H}_{A_i : C_i|B_i} X)$ with

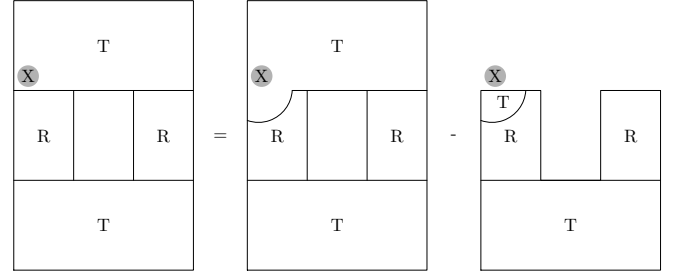


FIG. 1: After isolation move, conditional mutual spectrum is deformed in such a way that i) for the new conditional mutual spectrum, X is sufficiently far away from the reference party ii) the difference is a conditional mutual spectrum with small conditional mutual information.

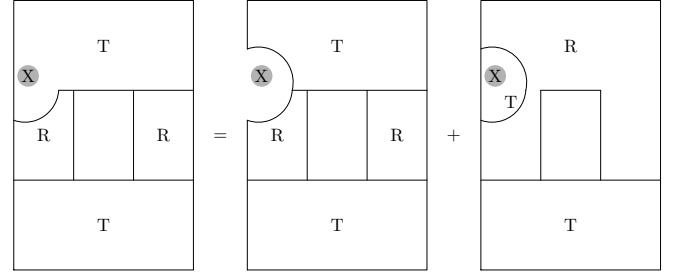


FIG. 2: After separation move, conditional mutual spectrum is deformed in such a way that i) for the new conditional mutual spectrum, X is sufficiently far away from both the reference and target party ii) the difference is a conditional mutual spectrum with small conditional mutual information.

$I(A_i : C_i|B_i) = o(1)$. To bound these terms, we introduce *absorption move*. See FIG.3.

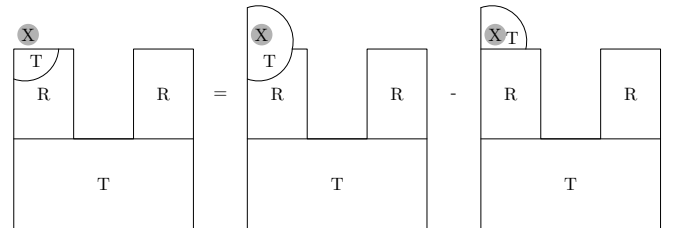


FIG. 3: Goal of absorption move is to change the correction terms into conditional mutual spectrum $\hat{H}_{A_i : C_i|B_i}$ such that i) support of X is contained in either A_i or C_i ii) $I(A_i : C_i|B_i)$ is small.

After applying the absorption move, corrections from the deformation move can be expressed as a sum of terms of the form $\text{Tr}(\rho_{A_i B_i C_i} \hat{H}_{A_i : C_i|B_i} X)$ with $X \subset A_i$. These terms can be bounded using the following lemma.

Lemma 1. *If Conjecture 1 is correct,*

$$|\text{Tr}_{ABC}(\rho_{ABC} \hat{H}_{A:C|B} O_A)| \leq \|O_A\| I(A : C|B), \quad (8)$$

where $\|\cdots\|$ is l_∞ norm.

Proof. Let $\mathcal{I}_A = \text{Tr}_{BC}(\rho_{ABC} \hat{H}_{A:C|B})$. $|\text{Tr}_A(\mathcal{I}_A O_A)| \leq \|O_A\| \|\mathcal{I}_A\|_1$. $|\cdot|_1$ is l_1 norm. Since \mathcal{I}_A is positive, $|\mathcal{I}_A|_1 = \text{Tr}_A \mathcal{I}_A = I(A : C|B)$. \square

To summarize, given a topologically nontrivial configuration, $\mathcal{C}(\hat{H}_{A:C|B}, X)$ can be expressed as $\mathcal{C}(\hat{H}_{A':C'|B'}, X)$ with $d(A'B'C', X) = O(l)$ and correction terms that can be expressed as sum of $\epsilon_{A_i} \|X\|$ and $\epsilon_{A_i:B_i} \|X\|$. Assuming i) X is localized in one of the original subsystems $A, B, C, D = (ABC)^c$ ii) each of the subsystems are sufficiently large, the correction terms vanish in $l \rightarrow \infty$ limit. One may be tempted to think that $\mathcal{C}(\hat{H}_{A':C'|B'}, X)$ vanishes in $l \rightarrow \infty$ limit as well, for correlation decays exponentially in the ground state of gapped system.[17, 18] While this statement is correct, we emphasize that modification of exponential clustering theorem is necessary.

Regularization of entanglement spectrum— Before we explain the details of our analysis, we would like to present a technical background about the subject. Exponential clustering theorem states that

$$|\mathcal{C}(O_A, O_B)| \leq c \|O_A\| \|O_B\| \min(|A|, |B|) e^{-\frac{d(A,B)}{\xi}} \quad (9)$$

for two spatially separated operator O_A and O_B , provided there is a gapped parent hamiltonian that consist of sum of geometrically local bounded-norm terms.[17, 18] Since spectrum of \hat{H}_A is formally unbounded, one cannot directly apply exponential clustering theorem. We circumvent this problem by regularizing the entanglement spectrum and bounding the error from the regularization procedure.

Definition 1. *Regularized entanglement spectrum \hat{H}_A^Λ with a cutoff Λ is*

$$\hat{H}_A^\Lambda = - \sum_{p \leq 1/\Lambda} \log p_i |i\rangle \langle i|. \quad (10)$$

Simple consequence of this construction is that l_∞ norm is bounded, i.e. $\|\hat{H}_A^\Lambda\| \leq \log \Lambda$. Correction from the regularization can be bounded using the following lemma.

Lemma 2.

$$\text{Tr}(\rho_{AB} \Delta_A^\Lambda O_B) \leq \|O_B\| \frac{\log \Lambda}{\Lambda} d_A \quad (11)$$

for $\Lambda \geq 2$, where $\Delta_A^\Lambda = \hat{H}_A - \hat{H}_A^\Lambda$.

Proof. Purify ρ_{AB} to $|\psi\rangle_{ABC}$. Rewrite the formula as $\text{Tr}(\rho_{AB} \Delta_A^\Lambda O_B) = \langle \psi |_{ABC} \Delta_A^\Lambda O_B | \psi \rangle_{ABC}$. Note that $|\psi\rangle_{ABC}$ admits a Schmidt decomposition $|\psi\rangle_{ABC} = \sum_i \sqrt{p_i} |i\rangle_A |i\rangle_{BC}$, where $\rho_A = \sum_i p_i |i\rangle_A \langle i|_A$. This in turn can be expressed as

$$\sum_{p_i \leq 1/\Lambda} -p_i \log p_i \langle i |_{BC} O_B | i \rangle_{BC}. \quad (12)$$

Using $-p_i \log p_i \leq \frac{1}{\Lambda} \log \Lambda$ and $|\langle i | O_B | i \rangle| \leq \|O_B\|$, one can complete the proof. \square

Interpretation — Setting $\Lambda = d_{ABC} e^{O(l)/\xi}$, we arrive at the following conclusion.

$$|\mathcal{C}(\hat{H}_{A:C|B}, X)| \leq \|X\| (\epsilon_1(l) + \epsilon_2(l)) l^2, \quad (13)$$

where ϵ_1 represents a deviation from ideal area law, and ϵ_2 represents an error from long range correlation. As $l \rightarrow \infty$, conditional mutual spectrum has vanishing correlation with any local operator, provided that i) X is supported on one of A, B, C , or D ii) both ϵ_1 and ϵ_2 decays sufficiently fast. In $l \rightarrow \infty$ limit, we have

$$\langle \hat{H}_{A:C|B} X \rangle = I(A : C|B) \langle X \rangle. \quad (14)$$

We conclude that operator $\hat{H}_{A:C|B}$ is topological, in a sense that i) it has vanishing correlation with any operator that is localized in one of the subsystems ii) its eigenvalues contain information about the phase. Set of assumptions to conclude so was i) correlation decays exponentially ii) extensive terms of entanglement entropy cancels out each other iii) deformation procedure separating X from ABC does not change the topology of the configuration.

We emphasize that derivation of our result is not necessarily restricted to the ground state. At finite temperature, entanglement entropy obtains volume contributions, but one may be able to show that those contributions can be canceled out as well. We expect these conditions to be met for quantum many-body systems at sufficiently high temperature.

In the large volume limit, it seems the local contribution of reduced density matrices cancel out each other, at least when $I(A : C|B) = o(\frac{1}{l^2})$. We do not have a definitive proof for this statement, but we argue as follows. If $\hat{H}_{A:C|B}$ contains a localized term, one could have chosen X to be an operator supported nearby so as to have large correlation with the local term. Such terms will violate eq.14. Our result suggests a decomposition of entanglement spectrum into i) terms that can be canceled out via suitable choice of subsystems ii) terms that cannot be canceled out and have small correlation with almost any local operators. It would be interesting if the terms of first kind can be shown to be quasilocal.

Evidence for the conjecture — We show that Conjecture 1 holds if reduced density matrices commute with each other. To see this, define $\omega_{ABC} = \rho_{AB} \rho_B^{-1} \rho_{BC}$. Using $D_1(\ln D_1 - \ln D_2) \geq D_1 - D_2$ for $D_1, D_2 \geq 0$, $[D_1, D_2] = 0$, we get

$$\rho_{ABC} (\ln \rho_{ABC} - \ln \omega_{ABC}) \geq \rho_{ABC} - \omega_{ABC}. \quad (15)$$

Since $\text{Tr}_C \omega_{ABC} = \rho_{AB}$, the conjecture holds even without taking a partial trace over B . [28] Important class of quantum states that satisfy the commutation relation is stabilizer quantum error correcting code, most notable example being Kitaev's toric code in this context. [19]

We have also numerically tested the conjecture for low-dimensional quantum states with dimensions $d_A = d_B =$

$d_C = 2, 3, 4$. For each choice of the dimensions, we have randomly generated 5×10^5 quantum states by i) randomly assigning the eigenvalues from uniform distribution over $[0, 1]$ and normalizing ii) applying a random unitary from Haar measure. We have also tested our conjecture to random pure states with same number of trials. In both cases, no counterexamples were found.

Outlook — We have presented a general argument as to why certain linear combination entanglement spectrum allows cancelation of local degrees of freedom, owing in part to an information-theoretic conjecture which might be interesting in its own right. While our formulation is not as precise as the ones described by variational wavefunctions,[10–12] it has an advantage of being applicable to general quantum states without a priori knowledge about the underlying structure other than approximate conditional independence and finite correlation length.

It would be interesting if the approximate conditional independence can be shown to hold in other systems. There are evidences suggesting that models based on BF theory should satisfy such condition[20], yet no studies have been performed for exotic models in 3D such as Haah’s code.[21] As for finite temperature states, approximate conditional independence is one of the key ideas of quantum belief propagation(QBP) algorithm.[22] Success of QBP indicates that our result may be applicable to finite temperature quantum states as well.[23]

On the other hand, we wish to find a deeper insight as to why conditional independence arises in these systems. In particular, exactly solvable models which satisfy exact conditional independence can be thought as a fixed point of some renormalization-group(RG) procedure.[24] Does conditional mutual information of topologically trivial configurations monotonically decreases under such RG flow?

We conclude with a remark that our correlation bound cannot be applied to operators that are supported on more than one of the subsystems. Ability to bound correlation of such form can be used for showing perturbative stability of topological entanglement entropy, but that shall be published elsewhere.[25]

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- [28] Taking a partial trace over C only, however, is not even hermitian in general. Even after hermitizing the operator, we found negative eigenvalues for some randomly generated samples.